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SOLVING DELAY DIFFERENTIAL EQUATIONS BY ABOODH TRANSFORMATION METHOD

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ABSTRACT

The solution for delay differential equations (DDEs) is achieved by implementing Aboodh transformation method and its properties on the given DDEs, with its unique formula to deal with non-linear terms and its high-level accuracy, but without recognizing restricted transformations, perturbation, linearization or discretization. According to the numerical evidences, the Aboodh transformation method converges favourably to the analytical solution. Maple 18 software is used for all computational frameworks..

KEYWORDS: *Aboodh Transforms, Delay Differential Equation, Approximate Solution, Partial Derivatives*

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1. INTRODUCTION

Delay differential equations are used in science and engineering, especially in the field of bioscience, physics, chemistry, population dynamics etc. Aboodh transformation method is used to solve DDEs of the type:

$$y^{(n)}(t) = f\big(t, y(t), y(t-\tau_i) \dots \dots y(t-\tau_i)\big), \qquad t \geq 0 \tag{1}$$

$$y(t) = g(t), \qquad -\tau \leq t \leq 0$$

Where $g(t)$ is the initial function, τ_i , $i > 0,$ is called the delay or lag function, f is a given function with $\tau_i \leq t.$ If $\tau_i > 0$ is a constant, it is a constant-dependent delay; if $\tau_i(x) \geq 0$ is time dependent, it is a time-dependent delay and if $u\big(\tau_i(x)\big) \geq 0$ is state dependent, it is a state-dependent delay.

The solution of DDEs has attracted the interest of many researchers in the recent days. Different numerical methods have been implemented and formulated for these equations, including adomian decomposition method [2-3], the variation iteration method [4], the differential transform method [5], the runge-kutta method [6], the hermite interpolation method [7], the variable multistep method [8], the decomposition method [9], the direct block one step method [10], B-spline collection method [11], the direct two and three point one-step block method [12], etc..

According to this study, the Aboodh transform method [13-16] is considered highly efficacious and dependable. The Aboodh transformation method is administered by addressing its properties on the given DDE where a well-posed formulae [13] are used to manipulate the nonlinear terms. The initial approximation and recursion formula are used to compute the components $u_{n+1}(x), n \geq 0,$ .The solution is approximately is written as the partial sum of the components $u_{n+1}(x), n \geq 0$. For N[AQ: N is not used in the components. Please check whether it is N or n.], where N is an integer. With high-level accuracy, the method can be applied easily by following the link: scitecresearch.com/journals/index.php/bjmp/article/download/1008/734. The Aboodh transformation method does not recognize restricted transformations, perturbation, linearization or discretization.

This article is organized as follows. Section 2 is on basic definitions and notations. Section 3 is about the Aboodh transform method for the components and nth order DDEs. Section 4 provides insight into numerical applications of the method to linear and nonlinear DDEs. Section 5 is on conclusion.

2. BASIC DEFINITIONS AND NOTATIONS

i. Let $f(x)$ for $x \geq 0,$ then the Aboodh transform ([13 – 16]) of $f(x)$ is a function of *s* defined by:

$$A[f(t)] = \frac{1}{s}\int_0^\infty f(x)e^{-sx}\,dx \qquad ,x \geq 0$$

ii. The DDE derivative from Aboodh transform [13] is achieved by part integration, that is

$$A[u'(x)] = r\,K(r) - \frac{u(0)}{r}$$

$$A[u''] = r^2K(r) - \frac{u'(0)}{r} - u(0)$$

Where

$$K_n(r) = r^nK(r) - \sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{r^{2-n+k}}$$

iii. References [13-16] provide some of the Aboodh transform properties, which are as follows:

a. $A[1] = \frac{1}{r^2}$

b. $A[x^n] = \frac{n!}{r^{n+2}}$

c. $A^{-1}\left[\frac{n!}{r^{n+2}}\right] = x^n$

iv. Table 1 lists the special functions and the standard Aboodh transform equivalent:

Table 1: Special Function versus Aboodh Transform Equivalent

Special Functions	Aboodh Transform Equivalent
$f(x)$	$A[f(x)] = K(r)$
$\sin ax$	$\frac{a}{r(r^2+a^2)}$
$\cos ax$	$\frac{1}{(r^2+a^2)}$
$\sinh ax$	$\frac{a}{r(r^2-a^2)}$
$\cosh ax$	$\frac{1}{(r^2-a^2)}$
e^{ax}	$\frac{1}{(r^2-ar)}$

3. ABOODH TRANSFORM METHOD

Consider the general nonlinear ordinary differential equation (ODE) of the form

$$\frac{d^m f(x)}{dx^m} + Rf(x) + Nf(x) = G(x), m = 1,2,3,4,\ldots \quad (2)$$

with initial condition

$$\left.\frac{d^{m-1} f(x)}{dx^{m-1}}\right|_{x=0} = g_{m-1}(x), m = 1,2,3,4,\ldots$$

$\frac{d^m f(x)}{dx^m}$ is the derivative of $f(x)$ of order m which is invertible, $Nf(x)$ is the nonlinear term, R is a linear operator and $G(x)$ is the source term. Applying the Aboodh transform ([13-16]), we obtain

$$A\left[\frac{d^m f(x)}{dx^m}\right] + A[Rf(x) + Nf(x)] = A[G(x)] \quad (3)$$

By definition (ii), we arrive at

$$A\left[\frac{d^m f(x)}{dx^m}\right] = A[G(x)] - A[Rf(x) + Nf(x)]$$

$$r^m K(r) - \sum_{n=0}^{m-1} \frac{1}{r^{2-m+n}} \frac{d^n f(0)}{dx^n} = A[G(x)] - A[Rf(x) + Nf(x)]$$

$$A[f(x)] = \sum_{n=0}^{m-1} \frac{1}{r^{2+n}} \frac{d^n f(0)}{dx^n} + \frac{1}{r^m} A[G(x)] - \frac{1}{r^m} A[Rf(x) + Nf(x)] \quad (4)$$

Applying the Aboodh inverse operator, A^{-1} on both sides of (4), we obtain

$$f(x) = A^{-1}\left[\sum_{n=0}^{m-1} \frac{1}{r^{2+n}} \frac{d^n f(0)}{dx^n}\right] + A^{-1}\left[\frac{1}{r^m} A[G(x)]\right] - A^{-1}\left[\frac{1}{r^m} A[Rf(x) + Nf(x)]\right] \quad (5)$$

where

$$\left.\frac{d^{m-1}f(x)}{dx^{m-1}}\right|_{x=0} = g_{m-1}(x), m = 1,2,3,4,\ldots$$

is the partial derivative of the initial condition.

Using the Aboodh transformation method, equation (5) can be written as

$$\sum_{n=0}^{\infty} f_n(x) = A^{-1}\left[\sum_{n=0}^{m-1}\frac{1}{r^{2+n}}\frac{d^n f(0)}{dx^n}\right] + A^{-1}\left[\frac{1}{r^m}A[G(x)]\right] - A^{-1}\left[\frac{1}{r^m}A[R\sum_{n=0}^{\infty} f_n(x) + \sum_{n=0}^{\infty} A_n(x)]\right] \quad (6)$$

where

$$A_n(x) = \sum_{r=0}^{n} f_r(x) f_{n-r}(x). \quad (7)$$

Comparing both sides of equation (6), we obtain

$$f_0(x) = A^{-1}\left[\sum_{n=0}^{m-1}\frac{1}{r^{2+n}}\frac{d^n f(0)}{dx^n}\right] + A^{-1}\left[\frac{1}{r^m}A[G(x)]\right]$$

$$f_1(x) = -A^{-1}\left[\frac{1}{r^m}A[Rf_0(x) + A_0(x)]\right]$$

$$f_2(x) = -A^{-1}\left[\frac{1}{r^m}A[Rf_1(x) + A_1(x)]\right]$$

$$\vdots$$

$$f_{n+1}(x) = -A^{-1}\left[\frac{1}{r^m}A[Rf_n(x) + A_n(x)]\right], m = 1,2,3,4,\ldots, n \geq 0 \quad (8)$$

Thus, the approximate solution can be written as

$$f(x) = \sum_{n=0}^{N} f_n(x), as\ N \to \infty. \quad (9)$$

Remark 3.1

In the case of nonlinear problems, equation (6) becomes

$$\sum_{n=0}^{\infty} f_n(x) = A^{-1}\left[\sum_{n=0}^{m-1}\frac{1}{r^{2+n}}\frac{d^n f(0)}{dx^n}\right] + A^{-1}\left[\frac{1}{r}A[G(x)]\right] - A^{-1}\left[\frac{1}{r}A[R\sum_{n=0}^{\infty} f_n(x) + \sum_{n=0}^{\infty} A_n(x)]\right]$$

Such that by comparison, the components $f_n(x)$ becomes

$$f_0(x) = A^{-1}\left[\sum_{n=0}^{m-1}\frac{1}{r^{2+n}}\frac{d^n f(0)}{dx^n}\right] + A^{-1}\left[\frac{1}{r}A[G(x)]\right]$$

$$f_1(x) = -A^{-1}\left[\frac{1}{r}A[Rf_0(x) + A_0(x)]\right]$$

$$f_2(x) = -A^{-1}\left[\frac{1}{r}A[Rf_1(x) + A_1(x)]\right]$$

$$\vdots$$

$$f_{n+1}(x) = -A^{-1}\left[\frac{1}{r}A[Rf_n(x) + A_n(x)]\right]$$

4. NUMERICAL APPLICATIONS

In numerical applications, the linear and nonlinear DDEs are resolved using the Aboodh transform method, and their results are compared using variation iteration with He as polynomials [4].

Example 4.1 [4]

Consider the following the first-order nonlinear delay differential equation (NDDE):

$$\frac{df}{dx} = 1 - 2f^2\left(\frac{x}{2}\right), 0 \le x \le 1, \tag{10}$$

With the initial condition

$$f(0) = 0.$$

The exact solution of the problem is

$$f(x) = \sin x$$

Solution

Applying the Aboodh transform on the both sides, we have

$$A\left[\frac{df(x)}{dx}\right] = A[1] - 2A\left[f^2\left(\frac{x}{2}\right)\right]$$

By definition (ii), we have

$$rK(r) - \frac{f(0)}{r} = \frac{1}{r^2} - 2A\left[f^2\left(\frac{x}{2}\right)\right]$$

$$A[f(x)] = \frac{1}{r^3} - \frac{2}{r}A\left[f^2\left(\frac{x}{2}\right)\right] \tag{11}$$

Applying the Aboodh inverse operator, A^{-1} on both sides of (11) we obtain

$$f(x) = A^{-1}\left[\frac{1}{r^3}\right] - 2A^{-1}\left[\frac{1}{r}A\left[f^2\left(\frac{x}{2}\right)\right]\right] \tag{12}$$

By definition (iiic), we have $A^{-1}\left[\frac{1}{r^3}\right] = x.$

Hence

$$f(x) = x - 2A^{-1}\left[\frac{1}{r}A\left[f^2\left(\frac{x}{2}\right)\right]\right] \quad \textbf{(13)}$$

By the Aboodh transform method, equation (13) can be written as

$$f_0(x) = x,$$

$$f_{n+1}(x) = -2A^{-1}\left[\frac{1}{r}A\left[f^2\left(\frac{x}{2}\right)\right]\right], n \geq 0, \quad \textbf{(14)}$$

where

$$A_n\left(\frac{x}{2}\right) = \sum_{r=0}^{n} f_r\left(\frac{x}{2}\right) f_{n-r}\left(\frac{x}{2}\right).$$

For $n = 0$, we have:

$$A_0\left(\frac{x}{2}\right) = \frac{x^2}{4}.$$

Which implies that

$$f_1(x) = -2A^{-1}\left[\frac{1}{r}A\left[\frac{x^2}{4}\right]\right] = -A^{-1}\left[\frac{1}{r^5}\right] = -\frac{x^3}{3!}.$$

For $n = 1$, we have:

$$A_1\left(\frac{x}{2}\right) = -\frac{x^4}{48}.$$

implying that

$$f_2(x) = \frac{1}{24}A^{-1}\left[\frac{1}{r}A[x^4]\right] = A^{-1}\left[\frac{1}{r^7}\right] = \frac{x^5}{5!}.$$

For $n = 2$, we have:

$$A_2\left(\frac{x}{2}\right) = \frac{x^6}{1440}.$$

Such that

$$f_3(x) = -\frac{1}{720}A^{-1}\left[\frac{1}{r}A[x^6]\right] = -A^{-1}\left[\frac{1}{r^9}\right] = -\frac{x^7}{7!}.$$

For $n = 3$, we have:

$$A_3\left(\frac{x}{2}\right) = -\frac{x^8}{80640}.$$

which gives

$$f_4(x) = \frac{1}{40320}A^{-1}\left[\frac{1}{r}A[x^8]\right] = A^{-1}\left[\frac{1}{r^{11}}\right] = \frac{x^9}{9!}.$$

Therefore, the approximate solution is given as

$$f(x) = \sum_{n=0}^{N} f_n(x) = f_0(x) + f_1(x) + f_2(x) + f_3(x) + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sin x$$

For $n \geq 0$, it is clear that the relative solution convenes quickly with the logical result. Using variation iteration method, the solution was obtained in [4] with He's polynomials.

Example 4.2

Consider the following the first-order linear delay differential equation (NDDE):

$$\frac{df(x)}{dx} = \frac{1}{2}\, e^{\frac{x}{2}} f\left(\frac{x}{2}\right) + \frac{1}{2}\, f(x), \qquad 0 \leq \mathrm{x} \leq 1, \tag{15}$$

With the initial condition

$$f(0) = 1.$$

The exact solution of the problem is

$$f(x) = e^x.$$

Solution

Applying the Aboodh transform on the both sides, we have:

$$A\left[\frac{df(x)}{dx}\right] = \frac{1}{2}\,\mathrm{A}\left[e^{0.5x} f\left(\frac{x}{2}\right) + f(x)\right]$$

By definition (ii) we have:

$$r\,K(r) - \frac{f(0)}{r} = \frac{1}{2}\,\mathrm{A}\left[e^{0.5x} f\left(\frac{x}{2}\right) + f(x)\right] \tag{16}$$

$$K(r) - \frac{1}{r^2} = \frac{1}{2}\,A\frac{1}{r}\left[e^{0.5x} f\left(\frac{x}{2}\right) + f(x)\right]$$

$$K(r) = \frac{1}{r^2} + \frac{1}{2}\,A\frac{1}{r}\left[e^{0.5x} f\left(\frac{x}{2}\right) + f(x)\right]$$

$$A[f(x)] = \frac{1}{r^2} + \frac{1}{2}\,A\frac{1}{r}\left[e^{0.5x} f\left(\frac{x}{2}\right) + f(x)\right]$$

$$f(x) = 1 + \frac{1}{2} A^{-1}\left[\frac{1}{r} A\left[e^{0.5x} f\left(\frac{x}{2}\right) + f(x)\right]\right] \quad (17)$$

By the Aboodh transform method, equation [17] can be written as:

$$f_0(x) = 1.$$

$$f_{n+1}(x) = \frac{1}{2} A^{-1}\left[\frac{1}{r} A\left[e^{0.5x} f_n\left(\frac{x}{2}\right) + f_n(x)\right]\right], \quad n \geq 0, \quad (18)$$

For $n = 0$,

$$f_1(x) = \frac{1}{2} A^{-1}\left[\frac{1}{r} A[e^{0.5x} + 1]\right],$$

$$= \frac{1}{2} A^{-1}\left[\frac{1}{r} A[e^{0.5x}] + \frac{1}{r} A[1]\right] \quad (19)$$

But

$$A[e^{0.5x}] = \frac{1}{r^2 - 0.5r} = \frac{1}{r^2}\left[\frac{1}{1 - \frac{0.5}{r}}\right]$$

$$= \frac{1}{r^2}\left[1 - \frac{0.5}{r}\right]^{-1}$$

$$= \frac{1}{r^2}\left[1 + \frac{0.5}{r} + \frac{0.25}{r^2} + \frac{0.125}{r^3} + \cdots\right]$$

Hence, equation (19) can be written as:

$$f_1(x) = \frac{1}{2} A^{-1}\left[\left[\frac{1}{r^3} + \frac{0.5}{r^4} + \frac{0.25}{r^5} + \frac{0.125}{r^6} + \cdots\right] + \frac{1}{r^3}\right]$$

By definition (iiic) we have: $f_1(x) = x + \frac{x^2}{8} + \frac{x^3}{48} + \frac{x^4}{384} + \cdots$

Using Maple 18 software and the above implementation, the following solution can be achieved for $n \geq 1$

$$f_2(x) = \frac{3}{8} x^2 + \frac{13}{192} x^3 + \frac{13}{1024} x^4 + \cdots$$

$$f_3(x) = \frac{5}{64} x^3 + \frac{63}{4096} x^4 + \cdots$$

Thus, the solution becomes

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = f_0(x) + f_1(x) + f_2(x) + f_3(x) + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$$

5. CONCLUSIONS

To arrive at a result for delay differential equations (DDEs), the Aboodh transformation method has been executed successfully. As shown in the examples, this method convenes quickly with the logical resolution. Therefore, the Aboodh transformation method is precise and added secure in pursuing the answer for DDEs.

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